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Hyper Pseudo BCK-Algebras with Conditions (S) and (P)

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ABSTRACT

In this paper, two conditions on hyper pseudo BCK-algebras, namely (S) and (P), are introduced and some related properties are investigated. Also, the notions of additive and Varlet hyper pseudo BCK-ideals of a hyper pseudo BCK-algebra are defined and the relations among them in hyper pseudo BCK-algebra with condition (S) are given.

Keywords: hyper pseudo, BCK-algebras, Varlet hyper pseudo.

1. INTRODUCTION

The study of BCK-algebras were initiated by Imai and Iseki (1966) as a generalization of the concept of set theoretic difference and propositional calculi. Pseudo BCK-algebras were introduced by Georgescu and Iorgulescu (2001) as a generalization of BCK-algebras in order to give a structure corresponding to pseudo MV-algebras, since the bounded commutative BCK-algebras correspond to MV-algebras. Hyper structure (called also multi algebras) was introduced by Marty (1934) at the 8th congress of Scandinavian Mathematicians. Since then many researchers have worked on algebraic hyper structures and developed them. Corsini and Leoreanu (2003), presented some of the numerous applications of algebraic hyper structure, especially those from last fifteen years, to the following subjects: geometry, hyper graphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, cods, median algebras, relation algebras, artificial intelligence and probabilities. Hyper structures have many applications to several sectors of both pure and applied sciences. Borzooei *et al.* (2000) applied the hyper structures to (pseudo) BCK-algebra and investigated some related properties. In this paper, we introduce two conditions (S) and (P) on hyper pseudo BCK-algebras and investigate some related properties. In particular, we prove that if a hyper pseudo BCK-algebra H satisfies the condition (P), it satisfies the conditionn (S), too. But the converse is not true in general. We define the product and union of hyper pseudo BCK-algebras and consider the condition (S) related to them. We also define the notions of additive and Varlet hyper pseudo BCK-ideal of a hyper pseudo BCK-algebra and give the relations among them in hyper pseudo BCK-algebra satisfying condition (S).

2. PRELIMINARIES

Definition 2.1. (Ggeorgesu and Iorgulescu (2001)).

A pseudo BCK-algebra is a structure $X=(X,*,\Diamond,0)$, where ``$*$'' and ``$\Diamond$'' are binary operations on X and ``0''is a constant element of X, that satisfies the following:

(a_1) $(x*y)\circ(x*z)\preceq z*y$, $(x\circ y)*(x\circ z)\ll z\circ y$,,
(a_2) $x*(x\circ y)\ll y$, $x\circ(x*y)\ll y$,
(a_3) $x\preceq x$,
(a_4) $0\preceq x$,
(a_5) $x\preceq y$, $y\preceq x$ implies $x=y$,
(a_6) $x\ll y \Leftrightarrow x*y=0 \Leftrightarrow x\circ y=0$.
for all $x,y,z\in X$.

Definition 2.2. (Borzooei, Rezazadeh and Ameri (2013)) .

A hyper pseudo BCK-algebra is a structure $(H;\circ,*,0)$ where ``$\circ$'' and``$*$'' are hyper operations on H and ``0''is a constant element that satisfies the following axioms:

(PHK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$, $(x * z) * (y * z) \ll x * y$,

(PHK2) $(x \circ y) * z = (x * z) \circ y$,

(PHK3) $x \circ y \ll x$, $x * y \ll x$,

(PHK4) $x \ll y$ and $y \ll x$ imply $x = y$,

For all $x, y, z \in H$, where $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

Definition 2.3. Let H be a hyper pseudo BCK-algebra and let S be a subset of H. If S is a hyper pseudo BCK-algebra with respect to both the hyperoperations ``$\circ$" and ``$*$" on H, then we say that S is a hyper subalgebra of H.

Proposition 2.4. (Borzooei, Rezazadeh and Ameri (2013)).

In any hyper pseudo BCK-algebra H, the following hold:

(i) $0 \circ 0 = 0$, $0 * 0 = 0$ $x \circ 0 = x$, $x * 0 = x$,

(ii) $0 \ll x$, $x \ll x$, $A \ll A$,

(iii) $0 \circ x = 0$, $0 * x = 0$, $0 \circ A = 0$, $0 * A = 0$,

(iv) $A \subseteq B$ implies $A \ll B$,

(v) $A \ll 0$ implies $A = \{0\}$,

(vi) $y \ll z$ implies $x \circ z \ll x \circ y$ and $x * z \ll x * y$,

(vii) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$, that is, $x \circ z \ll y \circ z$; $x * z = \{0\}$ implies $(x * z) * (y * z) = \{0\}$, that is, $x * z \ll y * z$,

(viii) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$ and $A * \{0\} = \{0\}$ implies $A = \{0\}$,

(ix) $(A \circ c) \circ (B \circ c) \ll A \circ B$, $(A * c) * (B * c) \ll A * B$ for all $x, y, z, c \in H$ and $A, B \subseteq H$.

Definition 2.5. (Borzooei, Rezazadeh and Ameri (2013)).

Let H be a hyper pseudo BCK-algebra. For any subset I of H and any element $y \in H$, we denote,

(1) $*(y, I)^{\ll} = \{x \in H \mid x * y \ll I\}$,

(2) $*(y, I)^{\subseteq} = \{x \in H \mid x * y \subseteq I\}$,

(3) $\circ(y,I)^{\ll} = \{x \in H \mid x \circ y \ll I\}$,

(4) $\circ(y,I)^{\subseteq} = \{x \in H \mid x \circ y \subseteq I\}$,

Definition 2.6. Let H be a hyper pseudo BCK-algebra. For any subset I of H and any element $y \in H$, we denote,

(1) $*(y,I)^{\cap} = \{x \in H \mid x * y \cap I \neq \varnothing\}$,

(2) $\circ(y,I)^{\cap} = \{x \in H \mid x \circ y \cap I \neq \varnothing\}$.

Definition 2.7. (Borzooei, Rezazadeh and Ameri (2013)).

Let H be a hyper pseudo BCK-algebra, $\varnothing \neq I \subseteq H$ and $0 \in I$. Then I is said to be a hyper pseudo BCK-ideal of

type (1), if $\forall y \in I, *(y,I)^{\ll} \subseteq I$ and $\circ(y,I)^{\blacklozenge} \subseteq I$;

type (2), if $\forall y \in I, *(y,I)^{\subseteq} I$ and $\circ(y,I)^{\ll} \subseteq I$;

type (3), if $\forall y \in I, *(y,I)^{\ll} \subseteq I$ and $\circ(y,I)^{\subseteq} \subseteq I$;

type (4), if $\forall y \in I, *(y,I)^{\subseteq} \subseteq I$ and $\circ(y,I)^{\subseteq} \subseteq I$;

type (5), if $\forall y \in I, *(y,I)^{\ll} \subseteq I$ or $\circ(y,I)^{\ll} \subseteq I$;

type (6), if $\forall y \in I, *(y,I)^{\subseteq} \subseteq I$ or $\circ(y,I)^{\ll} \subseteq I$;

type (7), if $\forall y \in I, *(y,I)^{\ll} \subseteq I$ or $\circ(y,I)^{\subseteq} \subseteq I$;

type (8),if $\forall y \in I, *(y,I)^{\subseteq} \subseteq I$ or $\circ(y,I)^{\subseteq} \subseteq I$;

type (9), if $\forall y \in I, *(y,I)^{\ll} \cap \circ(y,I)^{\ll} \subseteq I$;

type (10), if $\forall y \in I, *(y,I)^{\subseteq} \cap \circ(y,I)^{\ll} \subseteq I$;

type (11), if $\forall y \in I, *(y,I)^{\ll} \cap \circ(y,I)^{\subseteq} \subseteq I$;

type (12), if $\forall y \in I, *(y,I)^{\subseteq} \cap \circ(y,I)^{\subseteq} I$.

The relationships between all types of hyper pseudo BCK-ideals are given by the following diagram.

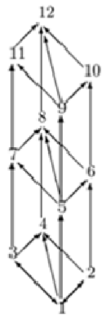

Figure 1

Definition 2.8. Let H be a hyper pseudo BCK algebra, $I \subseteq H$ and $0 \in I$. Then I is called a strong hyper pseudo BCK-ideal of H if for any $y \in I, *(y,I)^{\cap} \subseteq I$ and $\circ(y,I)^{\cap} \subseteq I$.

Theorem 2.9. *Let H be a hyper pseudo BCK-algebra and $I \subseteq H$. Then I is a strong hyper pseudo BCK-ideal of H if and only if the following hold:*

(i) $0 \in I$,
(ii) for any $y \in I, *(y,I)^{\cap} \subseteq I$ or for any $y \in I, \circ(y,I)^{\cap} \subseteq I$.

Definition 2.10. Let H be a hyper pseudo BCK-algebra and I be a subset of H. Then I is called reflexive if $x * x \subseteq I$ and $x \circ x \subseteq I$, for all $x \in H$.

Proposition 2.11. *Let H be a hyper pseudo BCK-algebra and $A \subseteq H$. If I is a hyper pseudo BCK-ideal of type 1,2,3,5 or 9 such that $A \ll I$, then $A \subseteq I$.*

Lemma 2.12. *Let H be a hyper pseudo BCK-algebra and I be a reflexive hyper pseudo BCK-ideal of type 1,2,3,5 or 9 of H. Then $(x * y) \cap I \neq \emptyset$ $((x \circ y) \cap I \neq \emptyset)$ implies $x * y \subseteq I (x \circ y \subseteq I)$.*

Definition 2.13. Let H be a hyper pseudo BCK -algebra and define the following subset of H :

$$S_*(H)=\{x\in H \mid x*x=\{0\}\}, S_\circ(H)=\{x\in H \mid x\circ x=\{0\}\}$$
$$S(H)=\{x\in H \mid x\circ x=x*x=\{0\}\}=S_\circ(H)\cap S_*(H).$$

Theorem 2.14.

(i) $S_*(H)$ *and* $S_\circ(H)$ *are closed subsets with respect to* ``$*$" *and* ``$\circ$", *respectively,*
(ii) *If* $S_*(H)=H$, *then* $x*y$ *is singleton, for all* $x,y\in H$,
(iii) *If* $S_\circ(H)=H$, *then* $x\circ y$ *is singleton, for all* $x,y\in H$,
(iv) *If* $S(H)=H$, *then* H *is a pseudo* BCK *-algebra.*

Definition 2.15. (Liu *et al.* (2007)).

A pseudo BCK algebra ``A" is called with condition $({}_P S)$ if the following condition holds:

$({}_P S)$ for all $x,y\in A$, $x\oplus y = max\{z \mid z*y \ll x\} = max\{z \mid z\circ x \ll y\}$ exists.

From now on, we let H be a hyper pseudo BCK -algebra, unless otherwise stated.

3. HYPER PSEUDO BCK -ALGEBRAS WITH CONDITION (S)

Definition 3.1. For any $a,b\in H$, , we denote

$$\Delta(a,b)=\{z\in H \mid 0\in(z*b)\circ a\}.$$

It is easy to see that $0,a,b\in\Delta(a,b)$. Moreover, by axiom (PHK2), we have

$$\Delta(a,b)=\{z\in H \mid 0\in(z\circ a)*b\}.$$

Note that, the equality $\Delta(a,b)=\Delta(b,a)$ is not true in general, as shown in the following example:

Example 3.2. Let $H=\{0,a,b,c\}$ and hyperoperations ``$*$'' and ``$\circ$'' on H be given by the following tables:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0,a,b\}$
c	$\{c\}$	$\{c,b\}$	$\{b\}$	$\{0,a,b\}$

$*$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$	$\{0,b\}$
c	$\{c\}$	$\{c\}$	$\{a,c\}$	$\{0,c\}$

Then $(H;\circ,*,0)$ is a hyper pseudo BCK-algebra. We can see that

$$\Delta(a,b)=\{0,a,b,c\}\neq\{0,a,b\}=\Delta(b,a).$$

Definition 3.3.

(i) Let $a,b\in H$. Then $u\in\Delta(a,b)$ is called the *greatest element* of $\Delta(a,b)$ and denoted by $a\oplus b$, if $x\ll u$ for all $x\in\Delta(a,b)$. In fact $a\oplus b=\max\{z:0\in(z*b)\circ a\}$.

(ii) H is called *with condition* (S), if for every $a,b\in H$, the set $\Delta(a,b)$ has the greatest element.

Example 3.4. Let $H=\{0,1,2,3,...\}$, the set of all non-negative integers with the natural ordering, and define the hyperoperations ``$*$'' and ``$\circ$'' on H as follows:

$$x*y=\begin{cases}\{0\} & \text{if } x<y\\ \{0,x\} & \text{if } x=y\\ \{x\} & \text{if} x>y\end{cases}\qquad x\circ y=\begin{cases}\{0,x\} & \text{if } x\le y\\ \{x\} & \text{if } x>y\end{cases}$$

Then $(H;\circ,*,0)$ is a hyper pseudo BCK-algebra and $\Delta(a,b)=\{z\mid z\le Max\{a,b\}\}$. Moreover,

$$a\oplus b=\begin{cases}b & \text{if } a\le b\\ a & \text{if } a>b.\end{cases}$$

Therefore, H is with condition (S).

Proposition 3.5. *Let* $a,b\in H$ *such that* $a\oplus b$ *exists. Then*

(i) $a\ll a\oplus b$ and $b\ll a\oplus b$,
(ii) $a\oplus 0=a=0\oplus a$.

Proof.

(i) The proof is straightforward.
(ii) Let $z\in \Delta(a,0)$. Then $0\in (z*0)\circ a$ and so by Proposition 2.5 (i), $0\in z\circ a$, that is, $z\ll a$. Now, since $a\in \Delta(a,0)$, it follows that $a\oplus 0=a$. Similarly, we can show that $0\oplus a=a$.

Let H be with condition (S). If $|H|\leq 3$, then for any $a,b\in H$, $\Delta(a,b)=\Delta(b,a)=H$ and so $a\oplus b=b\oplus a$. But if $|H|>3$, then $\oplus$ may not be commutative. To see this, let H be as in Example 3.2. Then $a\oplus b=c\neq b=b\oplus a$.

Proposition 3.7. *Let* H *be with condition* (S) *and* $x,y,z\in H$. *Then*

(i) $x\circ z\ll \bigcup_{\substack{u\in x\circ y\\ s\in y\circ z}} s\oplus u.$

(ii) $u\in x\circ y$ implies $x\ll y\oplus u$. Similarly, if $u\in x*y$, then $x\ll u\oplus y$.

Proof.

(i) Let $t\in x\circ z$ and $s\in y\circ z$. Since $(x\circ z)\circ(y\circ z)\ll x\circ y$, there exists $u\in x\circ y$ such that $0\in (t\circ s)*u$. It follows that, $t\in\Delta(s,u)$ and so $t\ll s\oplus u.$. Hence $x\circ z\ll \bigcup_{\substack{u\in x\circ y\\ s\in y\circ z}} s\oplus u.$

(ii) Let $u\in x\circ y$. Since $0\in u*u\subseteq (x\circ y)*u$, we get $x\in\Delta(y,u)$ and so $x\ll y\oplus u$. Similarly, we can show that, if $u\in x*y$, then $x\ll u\oplus y$.

Now we introduce some new hyper psuedo *BCK*-algebras and consider the condition (S) related to them.

Example 3.8. Let $(H;\circ,*,0)$ be a hyper pseudo BCK-algebra and $1\notin H$. We set $H^{'}=H\cup\{1\}$ and define the hyperoperations ``$*^{'}$'',``$\circ^{'}$'' on $H^{'}$ as follows:

$$x*^{'}y(x\circ^{'}y)=\begin{cases} x*y(x\circ y) & \text{if } x,y\in H\\ \{0\} & \text{if } x=y=1\\ \{0,x\} & \text{if } x\in H, y=1\\ \{1\} & \text{if } x=1, y\in H\end{cases}$$

Then it is easy to show that $(H^{'};*^{'},\circ^{'},0)$ is a bounded hyper pseudo BCK-algebra.

Definition 3.9. Let $(H;\circ,*,0)$ be a hyper pseudo BCK-algebra and $1\notin H$. Then the bounded hyper pseudo BCK-algebra $(H^{'}=H\cup\{1\};*^{'},\circ^{'},0)$ which is defined in Example 3.8 is called *the bounded Extention of* H.

Proposition 3.10. *If* H *is with condition* (S), *then* $H^{'}=H\cup\{1\}$, *the bounded Extention of* H, *is with condition* (S), *too.*

Proof. Let $a,b\in H$. We will show that $\Delta(a,b)$ has the greatest element. We need only to consider the two following cases:

(i) Let $a,b\in H$. By the definition of ``$*^{'}$'',``$\circ^{'}$'' on $H^{'}$, we have $(1*^{'}b)\circ^{'}a=1$. Hence $1\notin\Delta(a,b)$ and so $\Delta(a,b)\subseteq H$. Therefore $\Delta(a,b)$ has the greatest element.

(ii) Let $a=1$ or $b=1$. Then, since $0\in(z*^{'}b)\circ^{'}a$ for all $z\in H^{'}$, we get $\Delta(a,b)=H^{'}$ and so $a\oplus b=1$.

Theorem 3.11. *Let* $\{(H_i;*_i,\circ_i,0_i):i\in I\}$ *be a family of hyper pseudo* BCK*-algebras such that*

(i) $0_i=0$, for all $i\in I$,

(ii) $H_i\cap H_j=\{0\}$ *for all* $i,j\in I$ *with* $i\neq j$.

(iii) $H=\cup_{i\in I}H_i$

and define $x * y = \begin{cases} x *_i y, & \text{if } x, y \in H_i \\ \{x\}, & \text{otherwise} \end{cases}$ $\quad x \circ y = \begin{cases} x \circ_i y, & \text{if } x, y \in H_i \\ \{x\}, & \text{otherwise} \end{cases}$.

Then $(H;*,\circ,0)$ *is a hyper pseudo* BCK *-algebra, which is called the union hyper pseudo* BCK *-algebra.*

Proof. The proof is straightforward.

Theorem 3.12. *If* $H = H_1 \cup H_2$, *the union hyper pseudo* BCK *-algebra of* H_1 *and* H_2, *is with condition* (S), *then* $H_1 = \{0\}$ *or* $H_2 = \{0\}$.

Proof. Let $H_1 \neq \{0\}$ and $H_2 \neq \{0\}$, by the contrary. Then there are $0 \neq x \in H_1$ and $0 \neq y \in H_2$. Since $x \oplus y$ exists, we have $x \oplus y \in H_1$ or $x \oplus y \in H_2$. If $x \oplus y \in H_1$, then by Theorem 3.11, we get $((x \oplus y) * y) \circ x = (x \oplus y) \circ x$. Since $0 \in ((x \oplus y) * y) \circ x$, $x \oplus y \ll x$. Now, since $x \ll x \oplus y$, we obtain $x \oplus y = x$ and so from $y \ll x \oplus y$, we have $y \ll x$, that is, $0 \in y * x = \{y\}$. Therefore, $y = 0$, which is a contradiction. By a similar argument, we can show that if $x \oplus y \in H_2$, then $x = 0$, which is a contradiction, too. Therefore, H is not with condition (S).

Theorem 3.13. *Let* $(H_1;*_1,\circ_1,0_1)$ *and* $(H_2;*_2,\circ_2,0_2)$ *be two hyper pseudo* BCK *-algebrs and* $H = H_1 \times H_2$. *We define hyperoperations* ``$*$''*and*``$\circ$'' *and relation* ``$\ll$'' *on* H *as follows:*

$$(a_1,a_2) \circ (b_1,b_2) = (a_1 \circ_1 b_1, a_2 \circ_2 b_2) \text{ and } (a_1,a_2) * (b_1,b_2) = (a_1 *_1 b_1, a_2 *_2 b_2)$$
$$\text{and } (a_1,a_2) \ll (b_1,b_2) \Leftrightarrow a_1 \ll b_1, a_2 \ll b_2$$

for all $(a_1,a_2),(b_1,b_2) \in H$. *Then* $(H,\circ,*,0)$ *is a hyper pseudo* BCK *-algebra, which is called hyper pseudo product of* H_1 *and* H_2.

Proof. The proof is straightforward.

Theorem 3.14. *Let* $(H_1;\circ_1,*_1,0_1)$ *and* $(H_2;\circ_2,*_2,0_2)$ *be two hyper pseudo* BCK *-algebrs. Let* $H=H_1\times H_2$ *be hyper pseudo product of* H_1 *and* H_2*. Then,* H_1 *and* H_2 *are with condition* (S) *if and only if* H *is with condition* (S)*.*

Proof.
$(\Rightarrow)$ Let H_1 and H_2 be with condition (S), $(a_1,a_2),(b_1,b_2)\in H$ and $(z_1,z_2)\in\Delta((a_1,a_2),(b_1,b_2))$. Then $(0_1,0_2)\in((z_1,z_2)*(b_1,b_2))\circ(a_1,a_2)$ and so $0_1\in(z_1*_1 b_1)\circ_1 a_1$ and $0_2\in(z_2*_2 b_2)\circ_2 a_2$. That is, $z_1\in\Delta(a_1,b_1)$ and $z_2\in\Delta(a_2,b_2)$. Hence $(z_1,z_2)\ll(a_1\oplus b_1,a_2\oplus b_2)$. Now we show that $(a_1\oplus b_1,a_2\oplus b_2)\in\Delta((a_1,a_2),(b_1,b_2))$. Since $0_1\in((a_1\oplus b_1)*_1 b_1)\circ_1 a_1$ and $0_2\in((a_2\oplus b_2)*_2 b_2)\circ_2 a_2$, then
$(0_1,0_2)\in((a_1\oplus b_1,a_2\oplus b_2)*(b_1,b_2))\circ(a_1,a_2)$ and so $(a_1\oplus b_1,a_2\oplus b_2)\in\Delta((a_1,a_2),\,(b_1,b_2))$. Hence H is with condition (S).

$(\Leftarrow)$ Let $H=H_1\times H_2$ be with condition (S) and $x,y\in H_1$. Assume that $a=(x,0_2)$, $b=(y,0_2)$ and $a\oplus b=(a_1,a_2)$. Then
$(0_1,0_2)\in((a\oplus b)*b)\circ a=((a_1,a_2)*(y,0_2))\circ(x,0_2)$ and so $(0_1,0_2)\in((a_1*_1 y)\circ_1 x,(a_2*_2 0_2)\circ_2 0_2)$.

Hence, $0_1\in(a_1*_1 y)\circ_1 x$ and $0_2\in(a_2*0_2)\circ_2 0_2$. Then $a_1\in\Delta(x,y)$ and $a_2=0_2$. Now if $c\in\Delta(x,y)$, then $(c,0_2)\in\Delta(a,b)$ and so $(c,0_2)\ll a\oplus b$. Hence, $c\ll a_1$. Hence a_1 is the greatest hyper element of $\Delta(x,y)$. Therefore, H_1 is with condition (S). Similarly, we can show that H_2 is with condition (S), too.

Now, we define the notions of some new hyper pseudo BCK -ideals of a hyper pseudo BCK -algebra and give the relations among them in hyper pseudo BCK -algebra satisfying condition (S).

Definition 3.15. Let I be a nonempty subset of H. Then I is called a Varlet hyper pseudo BCK-ideal of H if

(VPHI1) $x \in I$ and $y \ll x$ imply $y \in I$,
(VPHI2) for any $x, y \in I$, there exists $z \in I$ such that $x \ll z$ and $y \ll z$.

Definition 3.16. Let H be with condition (S) and I be a non empty subset of H. Then I is called an additive hyper pseudo BCK-ideal of H if it satisfies (VPHI1) and (APHI) i.e., $x \in I$ and $y \in I$ imply $x \oplus y \in I$.

Example 3.17. Let $H = \{0, a, b, c\}$ and hyperoperations ``$\circ$'' and ``$*$'' on H be given by the following tables:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b\}$	$\{0\}$

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0,b\}$
c	$\{c\}$	$\{c\}$	$\{c,b\}$	$\{0,c\}$

Then $(H; \circ, *, 0)$ is a hyper pseudo BCK-algebra with condition (S). We can see that $I_1 := \{0, a, b\}$ is a Varlet hyper pseudo BCK-ideal and $I_2 := \{0, a\}$ is an additive hyper pseudo BCK-ideal.

Proposition 3.18. *Let H be with condition (S). Then every additive hyper pseudo BCK-ideal of H is a Varlet hyper pseudo BCK-ideal.*

Proof. It is a direct consequence of (VPHI1) and Proposition 3.5(i).

The following example shows that the converse of Proposition 3.18 is not true in general.

Example 3.19. Let H and I_1 be as in Example 3.17. Since $b \oplus b = \max\{z \mid 0 \in (z * b) \circ a\} = \max\{0, a, b, c\} = c$ and $c \notin I$, it follows that I_1 is not an additive hyper pseudo BCK-ideal.

Proposition 3.20. *Every Varlet hyper pseudo BCK-ideal of H is a hyper subalgebra of H.*

Proof. By (PHK3), the proof is straightforward.

Theorem 3.21. *Let H be with condition (S). Then every additive hyper pseudo BCK-ideal of H is a hyper pseudo BCK-ideal of any type 1,2,..,12.*

Proof. Let I be an additive hyper pseudo BCK-ideal of H. Since $0 \ll a$, for all $a \in H$ and $\varnothing \neq I$, we get $0 \in I$. Now let $y \in I$ and $x \in \circ(y,I)^{\ll}$. Then $x \circ y \ll I$ and so there exist $u \in x \circ y$ and $w \in I$ such that $u \ll w$. Hence by (VPHI1), we get $u \in I$. By Proposition 3.7, we have $x \ll y \oplus u$. Now, $u, y \in I$ implies that $y \oplus u \in I$ and so by (VPHI1), we have $x \in I$. Thus $\circ(y,I)^{\ll} \subseteq I$. Similarly, we obtain $*(y,I)^{\ll} \subseteq I$. Hence I is a hyper pseudo BCK-ideal of type 1. Therefore by Figure 1, it is of any type 1,2,...,12.

The following example shows that the converse of Proposition 3.20 is not true in general.

Example 3.22. Let $H = \{0,a,b,c,d\}$ and hyperoperations ``$\circ$" and ``$*$" on H be given by the following tables:

$\circ$	0	a	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
c	$\{c\}$	$\{c,b\}$	$\{b,d\}$	$\{0,a,b,d\}$	$\{b,d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{0,d\}$	$\{0,d\}$

$*$	0	a	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$	$\{0,b\}$	$\{0,a,b\}$
c	$\{c\}$	$\{c\}$	$\{a,c\}$	$\{0,c\}$	$\{a,b,c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{\circ,d\}$	$\{0,d\}$

Then $(H;*,\circ,0)$ is a hyper pseudo BCK-algebra with condition (S). We can see that $I := \{0,a,b\}$ is a hyper pseudo BCK-ideal of type 1 but it is not an additive hyper pseudo BCK-ideal, because $a,b \in I$ but $a \oplus b \notin I$.

Theorem 3.23. *In any hyper pseudo BCK algebra with condition (S), the concept of additive hyper pseudo BCK-ideal and strong hyper pseudo BCK-ideal are coincide.*

Proof. Let I be a strong hyper pseudo BCK-ideal of H and $x, y \in I$. Then by condition (S), $x \oplus y \in \Delta(x, y)$ and so $0 \in ((x \oplus y) \circ x) * y$. Hence there exists $t \in (x \oplus y) \circ x$ such that $0 \in t * y$. Thus $(t * y) \cap I \neq \varnothing$. Since I is a strong hyper pseudo BCK-ideal and $y \in I$, we get $t \in I$. Therefore $((x \oplus y) \circ x) \cap I \neq \varnothing$ and so by similar argument, we obtain $x \oplus y \in I$. Hence (APHI) holds. (VPHI1) follows from Proposition 2.11. Therefore I is an additive hyper pseudo BCK-ideal of H. Conversely, let I be an additive hyper pseudo BCK-ideal, $(x * y) \cap I \neq \varnothing$ and $y \in I$. Then, there is $t \in (x * y) \cap I$. Therefore $0 \in (x * y) \circ t$, that is, $x \in \Delta(t, y)$ and so $x \ll t \oplus y$. By $(APHI)$, we have $t \oplus y \in I$ and so by $(VPHI1)$, we get $x \in I$. Therefore, I is a strong hyper pseudo BCK-ideal.

Corollary 3.24. *Let H be with condition (S). Then every strong hyper pseudo BCK-ideal of H is a Varlet hyper pseudo BCK-ideal.*

Proof. It is a direct consequence of Proposition 3.18 and Theorem 3.23.

The following examples show that there is no relationship between two notions, Varlet hyper pseudo BCK-ideal and hyper pseudo BCK-ideal of type 1,2,...,12.

Example 3.25.

(i) Let $H = \{0, a, b, c\}$ and hyperoperations ``$\circ$'' and ``$*$'' on H be given by the following tables:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0\}$

$*$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0\}$

Then $(H; \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I := \{0, a, b\}$ is a hyper pseudo BCK-ideal of type 1 and so by Figure 1, it is any type. But it is not a Varlet hyper pseudo BCK-ideal because there is no $z \in I$ such that $a \ll z$ and $b \ll z$.

(i) Let $H=\{0,a,b\}$ and hyperoperations ``$\circ$" and ``$*$" on H be given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0,a\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0\}$

Then $(H,*,\circ,0)$ is a hyper pseudo BCK-algebra. We can see that $I:=\{0,a\}$ is a Varlet hyper pseudo BCK-ideal and it is not a hyper pseudo BCK-ideal of type 12 because $b\circ a=b*a\in I$ but $b\notin I$. Hence by Figure 1, I is not a hyper pseudo BCK-ideal of any type 1,..., 12.

Proposition 3.26. *If $S_*(H)=H$ or $S_\circ(H)=H$, then the relation ``$\ll$" on H is transitive.*

Proof. Let $x,y,z\in H$ such that $x\ll y$ and $y\ll z$. Then $0\in y*z$. Since $S_*(H)=H$, $y*z$ is singleton and so $y*z=\{0\}$. Now by (PHK1), we have $x*z=(x*z)*0=(x*z)*(y*z)\ll x*y=\{0\}$. Hence, $x*z=\{0\}$ and so $x\ll z$. Therefore, the relation ``$\ll$" is transitive. By a similar way, we can show that, if $S_\circ(H)=H$, then the the relation ``$\ll$" is transitive, too.

Theorem 3.27. *Let H be with condition (S). Then the following hold:*

(i) $(a\oplus b)*b\ll a$ for all $a,b\in H$ iff $S_*(H)=H$.

(ii) $(a\oplus b)\circ a\ll b$ for all $a,b\in H$ iff $S_\circ(H)=H$.

Proof.

(i) $(\Rightarrow)$ Let $a\in H$. By Proposition 3.5, $a=0\oplus a$. Then by hypothesis, we have $a*a=(0\oplus a)*a\ll 0$. This implies that $a*a=\{0\}$ and so $a\in S_*(H)$. Therefore $S_*(H)=H$.

$(\Leftarrow)$ Let $S_*(H)=H$ and $a,b\in H$. Then by Theorem 2.14, $((a\oplus b)*b)\circ a=((a\oplus b)\circ a)*b$ is singleton. On the other hand, by

Definition 3.3(i), $0 \in ((a \oplus b) * b) \circ a$. Therefore $((a \oplus b) * b) \circ a = \{0\}$ and so $(a \oplus b) * b \ll a$, for all $a, b \in H$.

(ii) The proof is similar to the proof of (i).

Corollary 3.28. *Let H be with condition (S). Then the following are equivalent:*

(i) *H is a pseudo BCK-algebra.*

(ii) *$(a \oplus b) \circ a \ll b$ and $(a \oplus b) * b \ll a$, for all $a, b \in H$.*

(iii) *$H = S(H)$.*

Proof. It is a direct consequence of Theorems 3.27 and 2.14.

Proposition 3.29. *Let H be with condition (S) and let $S_*(H) = H$ ($S_\circ(H) = H$). Then*

$$(\forall x, y, z \in H) \quad x \ll y \Rightarrow x \oplus z \ll y \oplus z \, (z \oplus x \ll z \oplus y).$$

Proof. Let $x, y \in H$ such that $x \ll y$. By Theorem 3.27, we have $(x \oplus z) * z \ll x$, for all $z \in H$ and so by transitivity of ``$\ll$" we get $(x \oplus z) * z \ll y$. Hence $0 \in ((x \oplus z) * z) \circ y$ and so $x \oplus z \in \Delta(y, z)$. This implies that $x \oplus z \ll y \oplus z$. Similarly, we can prove the second part of the proposition.

4. HYPER PSEUDO *BCK*-ALGEBRA WITH CONDITION (P)

The following example shows that the binary operation ``$\oplus$" on a hyper pseudo BCK-algebra with condition (S) may not be associative, in general.

Example 4.1. Let $H = \{0, a, b, c, d, e\}$ and hyperoperations ``$*$" and ``$\circ$" be given by the following tables:

$\circ$	0	a	b	c	d	e
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0,a,b\}$	$\{0,a,b\}$	$\{0,b\}$
c	$\{c\}$	$\{c,b\}$	$\{b,d\}$	$\{0,a,b,d\}$	$\{b,d\}$	$\{0,a,b,d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{0,d\}$	$\{0,d\}$	$\{0,d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e,d\}$	$\{e,d\}$	$\{0,e\}$

$*$	0	a	b	c	d	e
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$	$\{0,b\}$	$\{0,a,b\}$	$\{0,b\}$
c	$\{c\}$	$\{c\}$	$\{a,c\}$	$\{0,c\}$	$\{a,b,c\}$	$\{0,c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{0,d\}$	$\{0,d\}$	$\{0,d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e,d\}$	$\{e,d\}$	$\{0,e\}$

Then $(H;*,\circ,0)$ is a hyper pseudo BCK -algebra with condition (S). We can see that $(a \oplus b) \oplus c = c \oplus c = e$ and $a \oplus (b \oplus c) = a \oplus c = c.$ Hence $\oplus$ is not associative.

Now, we define another binary operation on H , which is associative.

Definition 4.2. We say that H is with condition (P) if there exists a binary operation $\odot$ on H such that

$$(P) \qquad 0 \in z \circ (x \odot y) \Leftrightarrow 0 \in (z * y) \circ x.$$

Example 4.3.

(i) Let $H = \{0, a, b\}$ and hyperoperations ``$\circ$'' and ``$*$'' and the binary operation ``$\odot$'' on H be given by the following tables:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

$*$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

$\odot$	0	a	b
0	0	a	b
a	a	b	b
b	b	b	b

Then it is routine to check that $(H;\circ,*,0)$ is a hyper pseudo BCK-algebra and $\odot$ satisfies condition (P).

(ii) Let H be with condition (P). Then $H'=H\cup\{1\}$, the bounded extension of H, is with condition (P). In fact, if $\odot$ is the binary operation on H satisfying condition (P) and $\odot'$ is defined on H' as follows:

$$x\odot' y=\begin{cases} x\odot y & \text{if } x,y\in H\\ 1 & \text{if } x=1 \text{ or } y=1\end{cases}$$

then one can check that $\odot'$ is a binary operation on H' which satisfies condition (P).

Lemma 4.4. *Let H be with condition (P). Then for all $x,y,z\in H$, we have*

(i) for all $a\in x*y$, $x\ll a\odot y$ and for all $a\in x\circ y$, $x\ll y\odot a$,
(ii) $0\in((y\odot x)*x)\circ y$ and $0\in((y\odot x)\circ y)*x)$,
(iii) if the relation $\ll$ is transitive, then, $x\ll y$ and $v\in y*z(v\in y\circ z)$ imply that there is $u\in x*z(u\in x\circ z)$ such that $u\ll v$.

Proof.

(i) For any $a\in x*y$, we have $0\in(x*y)\circ a$ and so by condition (P), $0\in x\circ(a\odot y)$. This implies that $x\ll a\odot y$. Similarly, we can show that $x\ll y\odot a$, for all $a\in x\circ y$.

(ii) Since $0\in(y\odot x)\circ(y\odot x)$, then by condition (P) the result holds.

(iii) Let $x,y,z,v\in H$ such that $x\ll y$ and $v\in y*z(v\in y\circ z)$. Then by (i), we get $y\ll v\odot z(y\ll z\odot v)$ and so by transitivity of $\ll$, we obtain $x\ll v\odot z(x\ll z\odot v)$. That is, $0\in x\circ(v\odot z)$ $(0\in x\circ(z\odot v)$. Hence by condition (P), $0\in(x*z)\circ v(0\in(x\circ z)*y)$. Therefore, there exists $u\in x*z(u\in x\circ z)$ such that $0\in u\circ v(0\in u*v)$ and so $u\ll v$.

Theorem 4.5. *If H is with condition (P), then H is with condition (S) and $x\odot y=x\oplus y$, for all $x,y\in H$.*

Proof. By Lemma 4.4(ii), we have $0 \in ((x \odot y) * y) \circ x$. This implies $x \odot y \in \Delta(x, y)$. Now let $z \in \Delta(x, y)$. Then $0 \in (z * y) \circ x$ and so by condition (P), $0 \in z \circ (x \odot y)$, that is, $z \ll x \odot y$. Hence $x \odot y$ is the greatest hyper element of $\Delta(x, y)$. Therefore H is with condition (S) and $x \odot y = x \oplus y$.

Corollary 4.6. *Let H be with condition (P). Then the binary operation $\odot$ on H satisfying condition (P) is unique.*

Proof. It is a direct consequence of Theorem 4.5 and the fact that the binary operation $\oplus$ is unique.

In the following example, we will show that conditions (P) and (S) are not equivalent, in general.

Example 4.7. Let $H = \{0, a, b, c, d\}$ be with condition (S) as in Example 3.22. If H is with condition (P), then $a \odot b = a \oplus b = c$. It is routine to check that $0 \notin (d * b) \circ a$. Since $0 \in d \circ c$, it follows from Definition 4.2 that $0 \in (d * b) \circ a$, which is a contradiction. Therefore H is not with condition (P).

Proposition 4.8. *$H = H_1 \times H_2$ is with condition (P) if and only if H_1 and H_2 are with condition (P).*

Proof. Let $(H_1; \circ_1, *_1, 0_1)$ and $(H_2; \circ_2, *_2, 0_2)$ be with condition (P). Define the binary operation $\odot$ on H as fallows:

$$(a_1, a_2) \odot (b_1, b_2) = (a_1 \odot_1 b_1, a_2 \odot_2 b_2), \tag{1}$$

where $\odot_1$ and $\odot_2$ are the binary operations on H_1 and H_2 satisfying condition (P), respectively. Clearly, $\odot$ is well-defined since $\odot_1$ and $\odot_2$ are well-defined. In order to show that $\odot$ satisfies condition (P) on H, let $z = (z_1, z_2), a = (a_1, a_2), b = (b_1, b_2) \in H$. Then, we have

$$\begin{aligned}0\in z\circ(a\odot b) &\Leftrightarrow 0_1\in z_1\circ_1(a_1\odot_1 b_1) \text{ and } 0_2\in z_2\circ_2(a_2\otimes_2 b_2), \text{ by (1)}\\ &\Leftrightarrow 0_1\in (z_1\circ_1 a_1)*_1 b_1) \text{ and } 0_2\in (z_2\circ_2 a_2)*_2 b_2,\\ &\quad \text{by condition } (P) \text{ on } H_1 \text{ and } H_2\\ &\Leftrightarrow (0_1,0_2)\in ((z_1\circ_1 a_1)*_1 b_1,(z_2\circ_2 a_2)*_2 b_2)\\ &\Leftrightarrow (0_1,0_2)\in ((z_1,z_2)\circ(a_1,a_2))*(b_1,b_2)\\ &\Leftrightarrow 0\in (z\circ a)*b.\end{aligned}$$

This implies that H is with condition (P). Conversely, we assume that H is with condition (P) and $x,y\in H_1$. Then, we can assume that $(x,0)\odot(y,0)=(a_1,a_2)$. for some $a_1\in H_1$ and $a_2\in H_2$. Define the binary operation $\odot_1$ on H_1 by $x\odot_1 y=a_1$. Clearly, $\odot_1$ is well-defined. Now, for every $z\in H_1$, , we have

$$\begin{aligned}0\in z\circ(x\odot_1 y) &\Leftrightarrow (0,0)\in (z\circ(x\odot_1 y),\{0\})\\ &\Leftrightarrow (0,0)\in (z,0)\circ((x\odot_1 y),0)\\ &\Leftrightarrow (0,0)\in (z,0)\circ(a_1,a_2)\\ &\Leftrightarrow (0,0)\in (z,0)\circ((x,0)\odot(y,0))\\ &\Leftrightarrow (0,0)\in ((z,0)\circ(x,0))*(y,0), \text{ by condition } (P) \text{ on } H\\ &\Leftrightarrow (0,0)\in ((z\circ x)*y,\{0\})\\ &\Leftrightarrow 0\in (z\circ x)*y.\end{aligned}$$

This implies that H_1 is with condition (P). By a similar way, we can show that H_2 is with condition (P), too.

Theorem 4.9. *Let $\odot$ be the binary operation on H satisfying condition (P). Then $(H;\odot,0)$ is a monoid.*

Proof. By Theorem 4.5, H is with condition (S) and $\odot=\oplus$. Also by Proposition 3.5 (ii), we have $a\oplus 0=0\oplus a=a$ for any $a\in H$. In order to show that $\odot$ is associative let $a\in\Delta(x\oplus y,z)$. Then $0\in(a*z)\circ(x\oplus y)$ and so there exists $t\in a*z$ such that $0\in t\circ(x\oplus y)$. Thus it follows from condition (P) that $0\in(t*y)\circ x$. Hence $0\in(t\circ x)*y$ and so $0\in((a*z)\circ x)*y$, that is, $0\in((a\circ x)*z)*y$. Therefore, there exists $t'\in a\circ x$ such that $0\in(t'*z)*y$ and so $0\in(t'*z)\circ y$. Then by condition (P), $0\in t'\circ(y\oplus z)$ and hence $0\in t'*(y\oplus z)$, that is, $0\in(a\circ x)*(y\oplus z)$.

Thus $a \in \Delta(x, y \oplus z)$ and so

$$\Delta(x \oplus y, z) \subseteq \Delta(x, y \oplus z). \qquad (2)$$

Now let $a \in \Delta(x, y \oplus z)$. Then $0 \in (a * (y \oplus z)) \circ x$ and so, $0 \in (a \circ x) * (y \oplus z)$. Hence there exists $t \in a \circ x$ such that $0 \in (t * (y \oplus z)$. Thus it follows from condition (P) that $0 \in (t * z) \circ y$ and so $0 \in ((a \circ x) * z) \circ y$. Hence by Definition 2.2, $0 \in ((a * z) \circ x) * y$ and so there exists $t' \in a * z$ such that $0 \in (t' \circ x) * y$, that is, $0 \in (t' * y) \circ x$. Therefore by condition (P), $0 \in t' \circ (x \oplus y)$ and so $0 \in (a * z) \circ (x \oplus y)$, that is, $a \in \Delta((x \oplus y), z)$. Hence

$$\Delta(x, y \oplus z) \subseteq \Delta(x \oplus y, z). \qquad (3)$$

From (2) and (3), we obtain $\Delta((x \oplus y), z) = \Delta(x, (y \oplus z))$ and so $(x \oplus y) \oplus z = x \oplus (y \oplus z)$. Therefore $(H; \odot, 0)$ is a monoid.

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